International Journal of Theoretical Physics, Vol. 31, No. 11, 1992

Quantum Field Theory of Binary Alternatives

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Received January 31, 1992

We describe a framework for quantum field theory that is based on quantized binary alternatives. We discuss the relation of the dynamics of quantum fields and the time evolution of the Heisenberg operators defined by the creation and annihilation operators of the alternatives. An explicit expression for the vacuum vector of the quantum fields is derived. Finally we discuss eigenstates of the momentum operators.

1. INTRODUCTION

In this paper we distinguish terminologically between abstract and concrete quantum theory. Abstract quantum theory is the theory of finite- or separably infinite-dimensional Hilbert space in which self-adjoint operators are called observables, and the metric defines probabilities for the prediction of measurement results. Concrete quantum theory offers the usual physical semantics to the observables under the three given concepts of position space, particles, and fields. From the outset we use these concepts in the frame of special relativity.

Historically the three guiding concepts belonged already to classical physics before quantum theory was discovered, and special relativity was formulated 20 years earlier than the mathematical frame of quantum mechanics. What we discuss is a logical inversion of this historical sequence. We maintain: Abstract quantum theory is sufficient for mathematically deducing the three-dimensional real position space, connected with time by the Poincaré group, as natural frame for expressing all possible versions of quantum theory. This follows from the fact that every Hilbert space can be described within the tensor product of two-dimensional complex metrical spaces V_2.

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0020-7748/92/1100-1929$06.50/0

The symmetry group of the two-dimensional complex space $U(2)$ and complex conjugation is sufficient for defining the symmetry of a real three-dimensional space, connected with a fourth (time) coordinate in the relativistic manner. A basis in such a V_2 can be defined by an operator with two eigenvalues. Logically, such an operator is just a binary alternative. In our physical use we call it, as an observable, an "ur" (from German Ur-Alternative = original alternative). The ur is just a bit of information. Abstract quantum theory thus is a theory on information, and the deduction of concrete quantum theory thus reduces "matter" to "information." But this philosophical remark is not the present topic. We study mathematical methods for deducing quantum field theory in detail from the ur theory.

What we later on called the "ur hypothesis" was introduced by v. Weisäcker (1955) and explicitly by Scheibe *et al.* (1958). Finkelstein (1969) independently developed an analogous theory on the same group-theoretic foundation under the title of "space-time-code." The mathematical frame which we use now was introduced by Castell (1975). It is described in v. Weisäcker (1985), Chapters 8–10. Abstract quantum theory is reconstructed from simple postulates by Drieschner *et al.* (1988). Now we are adding to this reconstruction its consequences in ur theory.

That abstract quantum theory can always be described in the tensor product of binary spaces is mathematically trivial. This statement must, however, be explained in three steps, of which only the two first are evidently trivial.

It is logically trivial that any n-fold logical alternative (n is a natural number or the set of natural numbers) can be decided by successively deciding a finite or infinite series of twofold ("binary") alternatives, i.e., yes–no decisions.

It can easily be mathematically proved that, as said before, any finite-dimensional or separable Hilbert space can be embedded into the tensor product of two-dimensional spaces.

At first sight it does not, however, seem evident that also any time dependence of the vectors in a Hilbert space can be derived from the time dependence of two-dimensional vectors, at least in a manner that would be describable by a meaningful Hamiltonian. In our early publications we offered this indeed only as a mathematical supposition. Yet now we propose the following argument.

In the Hamilton–Jacobi theory of classical mechanics, for any given Hamiltonian as a function of position and momentum, there exists always a canonical transformation into action and angle variables, i.e., into an inertial path. This transformation will indeed in general change the topology at large; else, e.g., a closed orbit could not be projected onto a straight line. If we accept this topological consequence, we will be able to transform any given dynamical law into any other. For the empirical interpretation we then select the form of the law whose topology at large agrees with our spatial intuition; then only the transformations of the Galilei group remain as permissible.

We can do a corresponding thing in quantum theory, including the definition of urs. The trivial separation of an n-fold alternative into urs is even logically not uniquely determined. Consider, as a simple example, a 4-fold alternative with the four possible answers a_i $(i=1, 2, 3, 4)$. It can be separated into 2-fold alternatives by combining two out of three different choices:

$$\begin{aligned} b_1&: \; i=1 \text{ or } 2; \qquad b_2: \; i=3 \text{ or } 4 \\ c_1&: \; i=1 \text{ or } 3; \qquad c_2: \; i=2 \text{ or } 4 \\ d_1&: \; i=1 \text{ or } 4; \qquad d_2: \; i=2 \text{ or } 3 \end{aligned} \tag{1}$$

Then, e.g., a_2 can be described by b_1c_2, by b_1d_2, or by c_2d_2. In the corresponding vector space $V_4=V_2\otimes V_2$ this choice is even continuously infinite. In a V_2 the original defining alternative A_2 can be rotated into any direction; the spin is the simple example. Hence we may describe all larger vector spaces by a continuum of possible definitions of the basic urs. We call this the "relativity of urs."

Now consider a Hilbert space with a given Hamiltonian H, transforming all vectors by e^{-iHt}. Then we select an arbitrary time-dependent redefinition of vectors and hence urs by e^{iHt}. It transforms the Hilbert space on a motionless one, and hence provides the measurement of transforming any dynamics in Hilbert space into any other. Trivially this will provide us with an extremely wild transformation of the topologies of the respective states. Then we select the "physical interpretation" by using a given semantics of the Hilbert space in concrete quantum theory, choosing that topology and hence dynamics which corresponds to our intuitive description of the position space.

In doing this we experience good luck. Defining urs in the simplest possible manner, that is, by assuming that in a transformation every ur is transformed by the same element of a given $U(2)$, we can deduce precisely the Poincaré group of ordinary special relativity, thus reconstructing the "intuitive" Minkowski space. We describe this in the manner introduced by Castell (1975), also described in v. Weisäcker (1985), p. 407.

The permissible transformations of an ur are $U(2)$ and complex conjugation. Castell chooses to represent complex conjugation by a linear transformation in a four-dimensional pseudo-Euclidean space V_4. This amounts to permitting the binary alternative of being an "ur" or an "anti-ur." The permissible transformations in this V_4 are $SU(2, 2)$, which is locally isomorphic to $SO(4, 2)$, the conformal group of special relativity. In the Fock space of the urs we can then build representations of $SO(4, 2)$ and hence of its subgroup, the Poincaré group. According to Wigner, its irreducible representations are particles. Thus the ur theory has the existence of particles as a necessary consequence.

Let it be clear, at this point, that the ur is by no means a "minute particle." Being one bit of information, a single ur, considered as a decision in, say, a finite universe, cannot distinguish more than one decision, say "up or down." Thus the single ur would be omnipresent in the universe. We would need roughly 10^{40} urs for locating an event within a distance of 10^{-12} cm on a line going through 10^{28} cm, as a plausible radius of a universe. Görnitz (1986, 1988) has shown that the information content of a nucleon, expressed in urs, corresponds exactly to the maximal entropy gain of a black hole by a nucleon falling into it, according to Bekenstein (1973) and Hawking (1975). In the ensuing sections the group representations are generated by bilinear combinations of production and annihilation operators of urs in their four states ($r = 1, 2$ representing an ur, $r = 3, 4$ representing an anti-ur) $a_r^\dagger$ and a_r.

We first assume the urs to have Bose statistics, i.e., the a_r, $a_r^\dagger$ to have the canonical commutation relations

$$[a_r, a_s^\dagger] = \delta_{rs}, \qquad [a_r, a_s] = [a_r^\dagger, a_s^\dagger] = 0 \tag{2}$$

We define the number operator

$$\nu_r = \tau_{rr} - \tfrac{1}{2}p \tag{3}$$

where we can use the following abbreviations:

$$\begin{aligned} \tau_{sr} &= \tfrac{1}{2}\{a_r, a_s^\dagger\} \\ \alpha_{sr}^\dagger &= \tfrac{1}{2}\{a_r^\dagger, a_s^\dagger\} \\ \alpha_{sr} &= \tfrac{1}{2}\{a_r, a_s\} \end{aligned} \tag{4}$$

and an operator s as half the difference between the numbers of urs and anti-urs,

$$s = \tfrac{1}{2}(n_1 + n_2 - n_3 - n_4) \tag{5}$$

Castell has shown that for these "Bose-urs" the Fock space of symmetric tensors contains precisely one representation for every value of s, which in Minkowski space means one massless particle with helicity s. He and his collaborators (Jacob, 1977; Heidenreich, 1981; Künemund, 1982, 1985) have shown that several massive and massless particles can be represented if we give the urs para-Bose statistics. Then p is the para-Bose order. The following sections study field and particle representations in this case. The formalism used there is the following.

The operators a_r, $a_r^\dagger$ must obey Green's commutation relations

$$\begin{aligned} \tfrac{1}{2}[\{a_r, a_s^\dagger\}, a_t] &= -\delta_{st}a_r \\ [\{a_r, a_s\}, a_t] &= [\{a_r^\dagger, a_s^\dagger\}, a_t^\dagger] = 0 \end{aligned} \tag{6}$$

This is achieved by defining Green's decomposition (Green, 1953)

$$a_r = \sum_{\alpha=1}^{p} b_r^\alpha, \qquad a_r^\dagger = \sum_{\alpha=1}^{p} b_r^{\dagger\alpha} \tag{7}$$

with

$$\begin{aligned} [b_r^\alpha, b_s^{\dagger\alpha}] &= \delta_{rs} \\ [b_r^\alpha, b_s^\alpha] &= [b_r^{\dagger\alpha}, b_s^{\dagger\alpha}] = 0 \\ \{b_r^\alpha, b_s^{\dagger\beta}\} = \{b_r^\alpha, b_s^\beta\} &= \{b_r^{\dagger\alpha}, b_s^\beta\} = 0 \qquad \text{for} \quad \alpha \neq \beta \end{aligned} \tag{8}$$

These bilinear operators enable us to construct the generators for the group $SU(2, 2)$ which is isomorphic to $SO(4, 2)$. The group $SO(4, 2)$ keeps invariant the quadratic form (v. Weisäcker, 1985, pp. 404–409) $G = x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2$. The group $SO(4, 2)$ has 15 generators, 7 compact M_{ik}, 8 noncompact N_{ik}:

$$\begin{aligned}
M_{12} &= \frac{i}{2}(+\nu_1 - \nu_2 + \nu_3 - \nu_4), & M_{15} &= \frac{i}{2}(+\tau_{12} + \tau_{21} - \tau_{34} - \tau_{43}) \\
M_{13} &= \frac{1}{2}(-\tau_{12} + \tau_{21} + \tau_{43} - \tau_{34}), & M_{25} &= \frac{1}{2}(+\tau_{12} - \tau_{21} + \tau_{43} - \tau_{34}) \\
M_{23} &= \frac{i}{2}(+\tau_{12} + \tau_{21} + \tau_{34} + \tau_{43}), & M_{35} &= \frac{i}{2}(\nu_1 - \nu_2 - \nu_3 + \nu_4) \\
M_{46} &= \frac{i}{2}(\nu + 2p) & & \\
N_{14} &= \frac{i}{2}(+\alpha_{13} + \alpha_{13}^\dagger - \alpha_{24} - \alpha_{24}^\dagger), & N_{16} &= \frac{1}{2}(-\alpha_{13} + \alpha_{13}^\dagger + \alpha_{24} - \alpha_{24}^\dagger) \\
N_{24} &= \frac{1}{2}(-\alpha_{13} + \alpha_{13}^\dagger - \alpha_{24} + \alpha_{24}^\dagger), & N_{26} &= \frac{i}{2}(-\alpha_{13} - \alpha_{13}^\dagger - \alpha_{24} - \alpha_{24}^\dagger) \\
N_{34} &= \frac{i}{2}(-\alpha_{14} - \alpha_{14}^\dagger - \alpha_{23} - \alpha_{23}^\dagger), & N_{36} &= \frac{1}{2}(+\alpha_{14} - \alpha_{14}^\dagger + \alpha_{23} - \alpha_{23}^\dagger) \\
N_{56} &= \frac{i}{2}(+\alpha_{14} + \alpha_{14}^\dagger - \alpha_{23} - \alpha_{23}^\dagger) & &
\end{aligned} \tag{9}$$

The group keeps constant the difference $2s$ of the number of urs and anti-urs. In the Poincaré group, M_{ik} ($i, k = 1, 2, 3$) are the components of angular momentum, N_{i4} $(i=1, 2, 3)$ generate Lorentz boosts, and the momenta P_μ $(\mu=0, 1, 2, 3)$ generating translations are defined by

$$\begin{aligned} P_i &= M_{i5}+N_{i6} \qquad (i=1, 2, 3) \\ P_0 &= M_{46}+N_{45} \end{aligned} \tag{10}$$

In the present paper we give an extensive description of free fields and particles. This will be useful in a later work on interaction.

2. FIELD THEORY

2.1. Decomposition of Relativistic Quantum Fields into Urs

We want to describe quantum field theory (QFT) within the framework of ur theory. To this aim, we describe how a quantum field (if it exists) may be decomposed as a sum over monomials in ur creation and annihilation operators a_i, $a_i^\dagger$, $i=1, \ldots, 4$. They fulfill the well-known commutation relations (CRs) of a para-Bose algebra (Green, 1953) of order p [the reason why para-Bose statistics is used is described elsewhere (Jacob, 1977; Heidenreich, 1981; Künemund, 1982)] and are represented by linear operators in a Fock space $\mathscr{F}_{\mathscr{A}}$. An operator A in $\mathscr{F}_{\mathscr{A}}$ may then be decomposed as (here and in the following we restrict ourselves to the Bose case $p=1$)

$$A=\sum_{I,J} c_{IJ}a^{\dagger I}a^J \tag{11}$$

where we use the short-hand notation

$$a^{\dagger I}=a_1^{\dagger I_1}a_2^{\dagger I_2}a_3^{\dagger I_3}a_4^{\dagger I_4}, \qquad a^J=a_1^{J_1}a_2^{J_2}a_3^{J_3}a_4^{J_4} \tag{12}$$

I and J are multi-indices. The complex coefficients c_{IJ} are uniquely determined by A.

If we assume that it is possible to describe quantum fields by ur theory, then the quantum fields $\Phi(x)$ should have a decomposition of the form (11), since $\Phi(x)$ is an operator in an underlying Hilbert space $\mathscr{H}$ to be identified with $\mathscr{F}_{\mathscr{A}}$.

Let us discuss the case of a Hermitian scalar field $\Phi(x)$, $x=(t, \mathbf{x})$. For every space-time point x a decomposition of the type (11) should be possible, where the coefficients will be x-dependent:

$$\Phi(x)=\sum_{I,J} c_{IJ}(x)a^{\dagger I}a^J \tag{13}$$

Since the field is Hermitian, we have the property that $c_{IJ}(x)=c_{JI}^*(x)$.

Given a relativistic QFT in Minkowski space, there is a unitary representation $U(\Lambda, b)$ of the Poincaré group, where Λ is a Lorentz transformation and b is a translation. Here we only consider the operators $U(b) = U(1, b)$ of translations. The transformation properties of a quantum field are

$$\Phi(x+b) = U(b)^\dagger \Phi(x) U(b) \tag{14}$$

where

$$U(b) = e^{-ib^\mu H_\mu} \tag{15}$$

and H_μ, $\mu = 0, \ldots, 3$, are the generators of space-time translations. In particular, H_0 is the Hamilton operator of the theory. The Heisenberg equations of motion for $\Phi(x)$ are

$$\partial_\mu \Phi(x) = i[H_\mu, \Phi(x)] \tag{16}$$

In ur theory, there seems to be a natural choice for the representation of the Poincaré group in the Fock space $\mathscr{F}_{\mathscr{A}}$ (Castell, 1975). The operators describing translations will then be identified with the Hamilton and momentum operators H_μ.

The formula (13) for $\Phi(x)$ is not helpful, since we do not know the x dependence of the $c_{IJ}(x)$. So we should try to learn something about the Poincaré transformation properties of (13).

2.2. Urs in the Heisenberg Picture

Let us apply the transformation (14) to formula (13):

$$\Phi(x+b) = U(b)^\dagger \Phi(x) U(b) = \sum_{I,J} c_{IJ}(x) U(b)^\dagger a^{\dagger I} a^J U(b) \tag{17}$$

We insert a unity $1 = U(b) U(b)^\dagger$ between the operators $a_i^\dagger$ and a_i:

$$\Phi(x+b) = \sum_{I,J} c_{IJ}(x) \prod_{k=1}^{4} (U(b)^\dagger a_k^\dagger U(b))^{I_k} \prod_{l=1}^{4} (U(b)^\dagger a_l U(b))^{J_l} \tag{18}$$

This formula motivates the definition of "space-time-dependent urs"

$$a_i(x) := U(x)^\dagger a_i U(x) \tag{19}$$

Formula (17) now reads

$$\Phi(x+b) = \sum_{I,J} c_{IJ}(x) a(b)^{\dagger I} a(b)^J \tag{20}$$

In particular, we have

$$\Phi(x) = \sum_{I,J} c_{IJ} a(x)^{\dagger I} a(x)^J, \qquad c_{IJ} := c_{IJ}(0) \tag{21}$$

The space-time dependence of Φ is now shifted from the coefficients $c_{IJ}(x)$ to the operators $a_i(x)$.

The $a_i(x)$ are nothing but the Heisenberg operators corresponding to the operators a_i. They satisfy the equation of motion

$$\partial_\mu a_i(x) = i[H_\mu, a_i(x)] \tag{22}$$

It turns out that (22) can be solved easily since the commutators of H_μ from v. Weisäcker (1985) and a_i are very simple. First of all, we introduce operators B_i by

$$\begin{aligned}
a_1 &= \frac{1}{\sqrt{2}}(B_1 + B_4)\\
a_2 &= \frac{1}{\sqrt{2}}(B_2 + B_3)\\
a_3 &= \frac{1}{\sqrt{2}}(B_2 - B_3)\\
a_4 &= \frac{1}{\sqrt{2}}(B_1 - B_4)
\end{aligned} \tag{23}$$

Since this transformation is a unitary transformation, the CRs of the B_is are the same as those of the a_is. The structure of the formulas is less complicated if they are expressed in terms of the B_i. Setting $x = (t, x^1, x^2, x^3)$, we obtain

$$\begin{aligned}
B_1(x) &= B_1 - \frac{i}{2}(t + x^3)(B_1 + iB_1^\dagger) - \frac{i}{2}x^1(B_3 + iB_3^\dagger) - \frac{1}{2}x^2(B_2 - iB_2^\dagger)\\
B_2(x) &= B_2 - \frac{i}{2}(t - x^3)(B_2 - iB_2^\dagger) - \frac{i}{2}x^1(B_4 - iB_4^\dagger) + \frac{1}{2}x^2(B_1 + iB_1^\dagger)\\
B_3(x) &= B_3 - \frac{i}{2}(t - x^3)(B_3 + iB_3^\dagger) - \frac{i}{2}x^1(B_1 + iB_1^\dagger) + \frac{1}{2}x^2(B_4 - iB_4^\dagger)\\
B_4(x) &= B_4 - \frac{i}{2}(t + x^3)(B_4 - iB_4^\dagger) - \frac{i}{2}x^1(B_2 - iB_2^\dagger) - \frac{1}{2}x^2(B_3 + iB_3^\dagger)
\end{aligned} \tag{24}$$

The translation of (24) to the operators $a_i(x)$ is simple and straightforward. The general form of $a_i(x)$ is

$$a_i(x) = a_i + x^\mu v_{i\mu} \tag{25}$$

Here $v_{i\mu}$ is a linear combination of the operators a_j, $a_j^\dagger$. The dependence of the $a_i(x)$ on x^μ is affine. If we define the light-cone variables x^+ and x^- by

$$x^+ := t + x^3, \qquad x^- := t - x^3 \tag{26}$$

we note that $B_1(x)$ and $B_4(x)$ depend only on (x^+, x^1, x^2) and $B_2(x)$ and $B_3(x)$ depend only on (x^-, x^1, x^2). The same is true for the pairs $a_1(x)$, $a_4(x)$ and $a_2(x)$, $a_3(x)$, respectively.

The space-time evolution of a quantum field (13) is now given by

$$\Phi(x) = \sum_{I,J} c_{IJ} \prod_{k=1}^{4} (a_k^\dagger + x^\mu v_{k\mu}^\dagger)^{I_k} \prod_{l=1}^{4} (a_l + x^\nu v_{l\nu})^{J_l} \tag{27}$$

Although the x^μ-dependence of $a_i(x)$ is next to trivial, $\Phi(x)$ can be expressed as a power series in x^μ with an arbitrarily complicated x^μ dependence. However, in general it is very difficult to rearrange the series because $v_{i\mu}$ contains annihilation as well as creation operators. Equation (27) shows that there could be a possibility to construct interacting quantum fields from simple building blocks (urs) with a simple space-time dependence (25).

The question arises if there is an interpretation of the x dependence of the a_i despite the fact that urs are not localized in space-time. Up to now, we assumed that the operators $U(\Lambda, b)$ describe active transformations. We suggest that the x dependence of the a_i comes from the transformation properties of the a_i with respect to passive transformations (of the observer). This means that urs depend on the frame of reference (or on the state of motion of the observer), which is no surprise since urs are bits of information, and a transformation in space and time of an object changes its description, the bits of information necessary to describe its properties. We could say that an ur is not Lorentz- or Poincaré-invariant, but it transforms covariantly.

2.3. Vacua in Ur Theory

In quantum theory and quantum field theory the vacuum vector $|0\rangle$ is important because it describes the ground state and contains information concerning the symmetries of the theory. Usually, $|0\rangle$ is also the state in the Fock space without any particles and quanta. In ur theory, the situation is slightly different. The state without any urs will be denoted by $|\Omega\rangle$ and is called the "ur vacuum" or the "logical vacuum." It is annihilated by the a_i:

$$a_i|\Omega\rangle = 0 \tag{28}$$

The Fock space is spanned by the basis vectors

$$|i_1 i_2 i_3 i_4\rangle := a_1^{\dagger i_1} a_2^{\dagger i_2} a_3^{\dagger i_3} a_4^{\dagger i_4} |\Omega\rangle \tag{29}$$

whose inner products in the para-Bose case can be calculated by a method based on polynomials associated with graphs (Graudenz, 1990).

It turns out that $|\Omega\rangle$ is not annihilated by the Hamilton operator H_0, and furthermore it is not an eigenvector of H_0. Therefore, it is not the ground state of the theory. This means that the vacuum $|\omega\rangle$ of the field theory is not $|\Omega\rangle$. One can calculate $|\omega\rangle$ as follows. We define operators P_j and Q_j by

$$\begin{aligned} B_1 &= \tfrac{1}{2}[(1+i)P_1+(1-i)Q_1] \\ B_2 &= \tfrac{1}{2}[(1-i)P_2-(1+i)Q_2] \\ B_3 &= \tfrac{1}{2}[(1+i)P_3+(1-i)Q_3] \\ B_4 &= \tfrac{1}{2}[(1-i)P_4-(1+i)Q_4] \end{aligned} \tag{30}$$

If we require the P_j and Q_j to be Hermitian, they are uniquely determined by $B_1, \ldots, B_4$. Their CRs are

$$[P_i, Q_j] = -i\delta_{ij} \tag{31}$$

We will call them formal coordinate and momentum operators because of (31). They have no relation to our space-time. Using these operators, the H_μ are given by

$$\begin{aligned} H_0 &= \tfrac{1}{2}(P_1^2+P_2^2+P_3^2+P_4^2) \\ H_1 &= \tfrac{1}{2}\{P_1, P_3\}+\tfrac{1}{2}\{P_2, P_4\} \\ H_2 &= \tfrac{1}{2}\{P_3, P_4\}-\tfrac{1}{2}\{P_1, P_2\} \\ H_3 &= \tfrac{1}{2}(P_1^2-P_2^2-P_3^2+P_4^2) \end{aligned} \tag{32}$$

In particular, H_0 is formally equivalent to the Hamilton operator of four free particles in one-dimensional space with momentum operators $P_1, \ldots, P_4$. In a coordinate representation, the Schrödinger wave function of the ground state of these particles is given by

$$\Psi_\omega(q_1, \ldots, q_4) = 1 \tag{33}$$

where the P_i and Q_i act as usual as differentiation and multiplication operators:

$$\begin{aligned} P_i\Psi(q_1, \ldots, q_4) &= -i\frac{\partial}{\partial q_i}\Psi(q_1, \ldots, q_4), \\ Q_i\Psi(q_1, \ldots, q_4) &= q_i\Psi(q_1, \ldots, q_4) \end{aligned} \tag{34}$$

$|\Omega\rangle$ is (up to a phase) uniquely determined by the condition

$$n|\Omega\rangle = 0 \tag{35}$$

where

$$n = \sum_{i=1}^{4} a_i^\dagger a_i = H_0' \tag{36}$$

is the number operator. In the coordinate representation (35) reads

$$H_0'\Psi_\Omega = 0 \tag{37}$$

where

$$H_0' = \tfrac{1}{2}(P_1^2 + P_2^2 + P_3^2 + P_4^2) + \tfrac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) - 4\tfrac{1}{2} \tag{38}$$

H_0' is the Hamilton operator of four uncoupled one-dimensional harmonic oscillators, whose ground state is given by

$$\Psi_\Omega(q_1, \ldots, q_4) = \left(\frac{1}{\pi^{1/4}}\right)^4 \exp\left[-\frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2)\right] \tag{39}$$

Using (33), we obtain

$$\Psi_\Omega = \left(\frac{1}{\pi^{1/4}}\right)^4 \left\{\exp\left[-\frac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2)\right]\right\}\Psi_\omega \tag{40}$$

Finally we arrive at

$$|\omega\rangle = (\pi^{1/4})^4\{\exp[\tfrac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2)]\}|\Omega\rangle \tag{41}$$

$|\omega\rangle$ is annihilated by all H_μ, which can be seen from (32)–(34). Therefore it is the vacuum of the field theory, since H_0 has lowest eigenvalue 0. However, $|\omega\rangle$ is not normalizable.

Field-theoretic investigations should be based on the "Lorentz vacuum" $|\omega\rangle$. This vacuum is not annihilated by the a_i, which could be rephrased as, "the vacuum of the field theory has a nontrivial logical structure."

3. PARTICLE STATES

3.1. Why Para-Bose Statistics for Urs?

An ur as a decidable binary alternative is defined as a concrete alternative having a definite meaning; therefore primarily we have Boltzmann statistics for the urs. To allow symmetry relations between them, which are a precondition for a definition of particles, the urs must become "more equal." So we define instead of different urs only different classes of urs which will be distinguished by an upper index. An ur may belong with an equal probability also to different classes.

Para-Bose statistics for the urs results if every ur can belong with equal chance to any class. This is described in more detail in Görnitz (1991). The number of different classes is called the order p of para-Bose statistics. The commutation relations for para-Bose creation and destruction operators for urs are given in (6). With them generators for representations of the Poincaré group can be constructed.

They are given in v. Weisäcker (1985) for the case of para-Bose statistics of order $p=1$ (i.e., Bose statistics). For the general case the generators are given in (9). The mathematical form of these operators was well known for a long time; however, no physical interpretation for them was given in the past. By the urs such an interpretation is introduced.

3.2. The Ur Vacuum and the Particle Vacuum

The ur vacuum or logical vacuum $|\Omega\rangle$ is defined as the state without any ur. For para-Bose creation and destruction operators $a_l^\dagger$ and a_k of order p the state $|\Omega\rangle$ fulfills

$$a_k|\Omega\rangle=0 \tag{42}$$

and

$$a_k a_l^\dagger|\Omega\rangle=p\delta_{kl}|\Omega\rangle, \qquad k, l=1, 2, 3, 4 \tag{43}$$

The ur vacuum is not invariant under the Poincaré group, e.g., for the momentum operators P_k we have

$$P_k|\Omega\rangle\neq 0, \qquad k=0, 1, 2, 3 \tag{44}$$

The Lorentz vacuum $|\omega\rangle$ for particle theory has the meaning that in this state no particle is present. This is a much larger amount of information than the knowledge "there is no ur." It can be expressed by urs only as an infinite sum in their creation operators. The Lorentz vacuum must be invariant under the Poincaré group and is defined therefore for any of its generators Q by the equations $Q|\omega\rangle=0$. They have the solution (41), or, written in the operators (4),

$$|\omega\rangle=\sum_\mu\sum_\lambda(-1)^{\mu+\lambda}i^{\mu-\lambda}(\mu!\,\lambda!)^{-1}\alpha_{14}^{\dagger\mu}\alpha_{23}^{\dagger\lambda}|\Omega\rangle=\{\exp[i(\alpha_{23}^\dagger-\alpha_{14}^\dagger)]\}|\Omega\rangle \tag{45}$$

The action of the creation and destruction operators of the urs on $|\omega\rangle$ is given in the Appendix.

3.3. States for Massless Particles

First of all we will construct states $\Phi^\dagger|\omega\rangle$ for massless particles. We define eigenstates of the momentum operators which fulfill the equations

$$P_k\Phi^\dagger|\omega\rangle=ip_k\Phi^\dagger|\omega\rangle \tag{46}$$

Note that we use anti-Hermitian P_k with the special choice for p_k:

$$(p_0-p_3)=p_1=p_2=0, \qquad p_0+p_3=\varepsilon_+\neq 0 \tag{47}$$

The general case can be obtained by transformations with generators of the Poincaré group.

For $\Phi^\dagger|\omega\rangle$ we make the ansatz

$$\Phi^\dagger|\omega\rangle=\sum_\sigma\sum_\mu c(\mu,\sigma)\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle \tag{48}$$

Since

$$[P_0-P_3,\alpha_{11}^\dagger]=0 \qquad \text{and} \qquad [P_0-P_3,\alpha_{14}^\dagger]=0 \tag{49}$$

we obtain

$$(P_0-P_3)\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle=0 \tag{50}$$

From

$$[P_1+iP_2,\alpha_{11}^{\dagger\sigma}]=0 \tag{51}$$

$$[P_1+iP_2,\alpha_{14}^{\dagger\mu}]|\omega\rangle=-\mu\alpha_{14}^{\dagger\mu-1}(i\alpha_{13}^\dagger-\tau_{12})|\omega\rangle=0 \tag{52}$$

we conclude

$$(P_1+iP_2)\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle=0 \tag{53}$$

and from

$$(P_1-iP_2)\alpha_{11}^{\dagger\sigma}=2\sigma\alpha_{11}^{\dagger\sigma-1}(i\alpha_{12}^\dagger-\tau_{13})+\alpha_{11}^{\dagger\sigma}(P_1-iP_2) \tag{54}$$

$$[P_1-iP_2,\alpha_{14}^{\dagger\mu}]=\mu\alpha_{14}^{\dagger\mu-1}(i\alpha_{24}^\dagger-\tau_{43}) \tag{55}$$

we get

$$\begin{aligned}(P_1-iP_2)\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}&=2\sigma\alpha_{11}^{\dagger\sigma-1}(i\alpha_{12}^\dagger-\tau_{13})\alpha_{14}^{\dagger\mu}+\alpha_{11}^{\dagger\sigma}(P_1-iP_2)\alpha_{14}^{\dagger\mu}\\&=2\sigma\alpha_{11}^{\dagger\sigma-1}\alpha_{14}^{\dagger\mu}(i\alpha_{12}^\dagger-\tau_{13})+\alpha_{11}^{\dagger\sigma}\mu\alpha_{14}^{\dagger\mu-1}(i\alpha_{24}^\dagger-\tau_{43})\\&\quad+\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}(P_1-iP_2)\end{aligned} \tag{56}$$

Therefore

$$(P_1-iP_2)\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle=0 \tag{57}$$

In addition we have

$$\begin{aligned}&(P_0+P_3-i\varepsilon_+)\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle\\&\quad=2\sigma\alpha_{11}^{\dagger\sigma-1}\tau_{14}\alpha_{14}^{\dagger\mu}|\omega\rangle+\alpha_{11}^{\dagger\sigma}(P_0+P_3+2i\sigma-i\varepsilon_+)\alpha_{14}^{\dagger\mu}|\omega\rangle\\&\quad=2\sigma\mu\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu-1}|\omega\rangle-i2\sigma\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle+(2i\sigma-i\varepsilon_+)\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle\\&\qquad+2i\mu\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle+\mu\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu-1}(\nu_1+\nu_4+p+\mu-1)|\omega\rangle\\&\quad=-i\varepsilon_+\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu}|\omega\rangle+\mu\alpha_{11}^{\dagger\sigma}\alpha_{14}^{\dagger\mu-1}(p+\mu-1+2\sigma)|\omega\rangle\end{aligned} \tag{58}$$

and get the conditions

$$-i\varepsilon_{+}c(\mu, \sigma)+(\mu+1)(p+\mu+2\sigma)c(\mu+1, \sigma)=0 \tag{59}$$

resp.

$$c(\mu+1, \sigma)=\frac{i\varepsilon_{+}c(\mu, \sigma)}{(\mu+1)(p+\mu+2\sigma)} \tag{60}$$

and therefore

$$c(\mu, \sigma)=c(0, \sigma)\frac{(i\varepsilon_{+})^{\mu}(p+2\sigma-1)!}{\mu!(p+2\sigma-1+\mu)!} \qquad (\text{for } \mu=0, 1, 2, \ldots) \tag{61}$$

In (61) we have no combination of different values for σ. Therefore in (48) the summation over σ can be omitted.

From

$$M_{12}|\Phi\rangle=\frac{i}{2}(n_1-n_2+n_3-n_4)|\Phi\rangle=\frac{i\sigma}{2} \tag{62}$$

it is obvious that a fixed σ is equivalent to a fixed spin.

A state of a massless particle with momentum in the $-z$ direction and helicity is given by

$$\Phi_{\alpha}^{\dagger}|\omega\rangle=c(0)\sum_{\mu}\frac{(i\varepsilon_{+})^{\mu}(p+1)!}{\mu!(p+1+\mu)!}a_{11}^{\dagger}a_{14}^{\dagger\mu}|\omega\rangle \tag{63}$$

and for the opposite helicity by

$$\Phi_{\beta}^{\dagger}|\omega\rangle=c(0)\sum_{\mu}\frac{(i\varepsilon_{+})^{\mu}(p+1)!}{\mu!(p+1+\mu)!}a_{44}^{\dagger}a_{14}^{\dagger\mu}|\omega\rangle \tag{64}$$

3.4. States for Spinless Particles

For massive bosons without spin (i.e., $n_1+n_3=n_2+n_4$) we make an ansatz

$$\Phi^{\dagger}|\omega\rangle=\sum_{\sigma}\sum_{\mu}\sum_{\lambda}h(\mu, \lambda, \sigma)a_{13}^{\dagger\sigma}a_{24}^{\dagger\sigma}a_{14}^{\dagger\mu}a_{23}^{\dagger\lambda}|\omega\rangle \tag{65}$$

Acting with P_1+iP_2 on each of the monomials, we obtain

$$\begin{aligned}(P_1+iP_2)&a_{13}^{\dagger\sigma}a_{24}^{\dagger\sigma}a_{14}^{\dagger\mu}a_{23}^{\dagger\lambda}|\omega\rangle\\ &=\mu\lambda a_{13}^{\dagger\sigma+1}a_{24}^{\dagger\sigma}a_{14}^{\dagger\mu-1}a_{23}^{\dagger\lambda-1}|\omega\rangle\\ &\quad+\sigma(\mu+\lambda+p+\sigma-1)a_{13}^{\dagger\sigma}a_{24}^{\dagger\sigma-1}a_{14}^{\dagger\mu}a_{23}^{\dagger\lambda}|\omega\rangle\end{aligned} \tag{66}$$

From

$$(P_1 + iP_2)\Phi^\dagger|\omega\rangle = 0 \tag{67}$$

the condition

$$\begin{aligned}(\mu+1)(\lambda+1)h(\mu+1, \lambda+1, \sigma)\\ +(\sigma+1)(\mu+\lambda+p+\sigma)h(\mu, \lambda, \sigma+1) = 0\end{aligned} \tag{68}$$

follows. Furthermore we have

$$\begin{aligned}&(P_1 - iP_2)a_{13}^{\dagger\sigma} a_{24}^{\dagger\sigma} a_{14}^{\dagger\mu} a_{23}^{\dagger\lambda}|\omega\rangle\\ &\quad = [-\mu\lambda a_{13}^{\dagger\sigma} a_{24}^{\dagger\sigma+1} a_{14}^{\dagger\mu-1} a_{23}^{\dagger\lambda-1}\\ &\qquad -\sigma(\mu+\lambda+p+\sigma-1)a_{24}^{\dagger\sigma} a_{13}^{\dagger\sigma-1} a_{14}^{\dagger\mu} a_{23}^{\dagger\lambda}]|\omega\rangle\end{aligned} \tag{69}$$

Thus, from

$$(P_1 + iP_2)\Phi^\dagger|\omega\rangle = 0 \tag{70}$$

the same condition results:

$$\begin{aligned}-(\mu+1)(\lambda+1)h(\mu+1, \lambda+1, \sigma)\\ -(\sigma+1)(\mu+\lambda+p+\sigma)h(\mu, \lambda, \sigma+1) = 0\end{aligned} \tag{71}$$

Next, we have

$$\begin{aligned}&(P_4 + P_3 - im_+)a_{13}^{\dagger\sigma} a_{24}^{\dagger\sigma} a_{14}^{\dagger\mu} a_{23}^{\dagger\lambda}|\omega\rangle\\ &\quad = (-im_+ a_{13}^{\dagger\sigma} a_{24}^{\dagger\sigma} a_{14}^{\dagger\mu} a_{23}^{\dagger\lambda} + \sigma^2 a_{13}^{\dagger\sigma-1} a_{24}^{\dagger\sigma-1} a_{14}^{\dagger\mu} a_{23}^{\dagger\lambda+1})|\omega\rangle\\ &\qquad +\mu(p+\mu-1+2\sigma)a_{13}^{\dagger\sigma} a_{24}^{\dagger\sigma} a_{23}^{\dagger\lambda} a_{14}^{\dagger\mu-1}|\omega\rangle\end{aligned} \tag{72}$$

From

$$(P_4 + P_3 - im_+)\Phi^\dagger|\omega\rangle = 0 \tag{73}$$

we conclude that the condition

$$\begin{aligned}(-im_+)h(\mu, \lambda, \sigma) + (\sigma+1)^2 h(\mu, \lambda-1, \sigma+1)\\ +(\mu+1)(p+\mu+2\sigma)h(\mu+1, \lambda, \sigma) = 0\end{aligned} \tag{74}$$

has to hold. From

$$\begin{aligned}&(P_4 - P_3 - im_-)a_{13}^{\dagger\sigma} a_{24}^{\dagger\sigma} a_{14}^{\dagger\mu} a_{23}^{\dagger\lambda}|\omega\rangle\\ &\quad = -im_- a_{13}^{\dagger\sigma} a_{24}^{\dagger\sigma} a_{14}^{\dagger\mu} a_{23}^{\dagger\lambda}|\omega\rangle - \sigma^2 a_{13}^{\dagger\sigma-1} a_{24}^{\dagger\sigma-1} a_{14}^{\dagger\mu+1} a_{23}^{\dagger\lambda}|\omega\rangle\\ &\qquad -\lambda(p+\lambda-1+2\sigma)a_{13}^{\dagger\sigma} a_{24}^{\dagger\sigma} a_{14}^{\dagger\mu} a_{23}^{\dagger\lambda-1}|\omega\rangle\end{aligned} \tag{75}$$

and

$$(P_4 - P_3 - im_-)\Phi^\dagger|\omega\rangle = 0 \tag{76}$$

we obtain the condition

$$\begin{aligned}(-im_-)h(\mu, \lambda, \sigma) - (\sigma+1)^2 h(\mu-1, \lambda, \sigma+1)\\ -(\lambda+1)(p+\lambda+2\sigma)h(\mu, \lambda+1, \sigma) = 0\end{aligned} \tag{77}$$

Written in another form, the result is

$$\begin{aligned}-(\mu+1)(\lambda+1)h(\mu+1, \lambda+1, \sigma)\\ =(\sigma+1)(\lambda+\mu+p+\sigma)h(\mu, \lambda, \sigma+1)\end{aligned} \tag{78}$$

$$\begin{aligned}im_+h(\mu, \lambda+1, \sigma) - (\mu+1)(p+\mu+2\sigma)h(\mu+1, \lambda+1, \sigma)\\ =(\sigma+1)^2 h(\mu, \lambda, \sigma+1)\end{aligned} \tag{79}$$

$$\begin{aligned}-im_-h(\mu+1, \lambda, \sigma) - (\lambda+1)(p+\lambda+2\sigma)h(\mu+1, \lambda+1, \sigma)\\ +(\sigma+1)^2 h(\mu, \lambda, \sigma+1)\end{aligned} \tag{80}$$

Combining conditions (79) and (80), we obtain

$$\begin{aligned}im_+h(\mu, \lambda+1, \sigma) - (\mu+1)(p+\mu+2\sigma)h(\mu+1, \lambda+1, \sigma)\\ = -im_-h(\mu+1, \lambda, \sigma) - (\lambda+1)(p+\lambda+2\sigma)h(\mu+1, \lambda+1, \sigma)\end{aligned} \tag{81}$$

and therefore

$$\begin{aligned}im_+h(\mu, \lambda+1, \sigma) + im_-h(\mu+1, \lambda, \sigma)\\ =(p+\mu+\lambda+2\sigma+1)(\mu-\lambda)h(\mu+1, \lambda+1, \sigma)\end{aligned} \tag{82}$$

From (78) and (79) after multiplication with $(\sigma+1)$, resp. $(\lambda+\mu+p+\sigma)$, we get

$$\begin{aligned}-(\sigma+1)(\mu+1)(\lambda+1)h(\mu+1, \lambda+1, \sigma)\\ =(\sigma+1)^2(\lambda+\mu+p+\sigma)h(\mu, \lambda, \sigma+1)\end{aligned} \tag{83}$$

and

$$\begin{aligned}(\lambda+\mu+p+\sigma)[+im_+h(\mu, \lambda+1, \sigma) - (\mu+1)(p+\mu+2\sigma)h(\mu+1, \lambda+1, \sigma)]\\ =(\sigma+1)^2(\lambda+\mu+p+\sigma)h(\mu, \lambda, \sigma+1)\end{aligned} \tag{84}$$

i.e.,

$$\begin{aligned}(\mu+1)[(\lambda+\mu+p+\sigma)(p+\mu+2\sigma) - (\sigma+1)(\lambda+1)]h(\mu+1, \lambda+1, \sigma)\\ =(\lambda+\mu+p+\sigma)im_+h(\mu, \lambda+1, \sigma)\end{aligned} \tag{85}$$

And from (78) and (80) after multiplication with $(\sigma+1)$, resp. $(\lambda+\mu+p+\sigma)$, we get

$$\begin{aligned}
&-(\sigma+1)(\mu+1)(\lambda+1)h(\mu+1,\lambda+1,\sigma)\\
&\quad=(\sigma+1)^2(\lambda+\mu+p+\sigma)h(\mu,\lambda,\sigma+1)(\lambda+\mu+p+\sigma)\\
&\qquad\times[-im_-h(\mu+1,\lambda,\sigma)-(\lambda+1)(p+\lambda+2\sigma)h(\mu+1,\lambda+1,\sigma)]\\
&\quad=(\sigma+1)^2(\lambda+\mu+p+\sigma)h(\mu,\lambda,\sigma+1)
\end{aligned}\tag{86}$$

Therefore

$$\begin{aligned}
&(\lambda+1)[(\lambda+\mu+p+\sigma)(p+\lambda+2\sigma)-(\sigma+1)(\mu+1)]h(\mu+1,\lambda+1,\sigma)\\
&\quad=-im_-(\lambda+\mu+p+\sigma)h(\mu+1,\lambda,\sigma)
\end{aligned}\tag{87}$$

For $\mu>0$, $\lambda>0$ and after renaming $\mu\to\mu-1$, resp. $\lambda\to\lambda-1$, we get

$$h(\mu+1,\lambda,\sigma)=\frac{im_+(\lambda+\mu+p+\sigma-1)h(\mu,\lambda,\sigma)}{(\mu+1)(\lambda+\mu+p+\sigma-1)(p+\mu+2\sigma)-\lambda(\sigma+1)}\quad\text{for}\quad\lambda>0\tag{88}$$

$$h(\mu,\lambda+1,\sigma)=\frac{(-im_-)(\lambda+\mu+p+\sigma-1)h(\mu,\lambda,\sigma)}{(\lambda+1)(\lambda+\mu+p+\sigma-1)(p+\lambda+2\sigma)-\mu(\sigma+1)}\quad\text{for}\quad\mu>0\tag{89}$$

Because of

$$\begin{aligned}
&[(\lambda+\mu+p+\sigma-1)(p+\mu+2\sigma)-\lambda(\sigma+1)]\\
&\quad=[(\mu+p+\sigma-1)(p+\mu+\lambda+2\sigma)]\\
&\qquad\times[(\lambda+\mu+p+\sigma-1)(p+\lambda+2\sigma)-\mu(\sigma+1)]\\
&\quad=[(\lambda+p+\sigma-1)(p+\lambda+\mu+2\sigma)]
\end{aligned}\tag{90}$$

we obtain for (88) and (89)

$$\begin{aligned}
h(\mu+1,\lambda,\sigma)&=h(\mu,\lambda,\sigma)\frac{-im_+(\lambda+\mu+p+\sigma-1)}{(\mu+1)(\mu+p+\sigma-1)(p+\mu+\lambda+2\sigma)}\quad\text{for}\quad\lambda>0\\
h(\mu,\lambda+1,\sigma)&=h(\mu,\lambda,\sigma)\frac{-im_-(\lambda+\mu+p+\sigma-1)}{(\lambda+1)(\lambda+p+\sigma-1)(p+\lambda+\mu+2\sigma)}\quad\text{for}\quad\mu>0
\end{aligned}\tag{91}$$

Another equation results from (79) and (80). We get from (79) after multiplication with $(\lambda+1)(p+\lambda+2\sigma)$

$$
\begin{aligned}
&(\lambda+1)(p+\lambda+2\sigma)(\sigma+1)^2 h(\mu, \lambda, \sigma+1) \\
&\quad -(\lambda+1)(p+\lambda+2\sigma) i m_+ h(\mu, \lambda+1, \sigma) \\
&\quad = -(\lambda+1)(p+\lambda+2\sigma)(\mu+1)(p+\mu+2\sigma) h(\mu+1, \lambda+1, \sigma)
\end{aligned} \tag{92}
$$

and furthermore from (80) after multiplication with $(\mu+1)(p+\mu+2\sigma)$

$$
\begin{aligned}
&(\mu+1)(p+\mu+2\sigma)(\sigma+1)^2 h(\mu, \lambda, \sigma+1) \\
&\quad +(\mu+1)(p+\mu+2\sigma) i m_- h(\mu+1, \lambda, \sigma) \\
&\quad = -(\mu+1)(p+\mu+2\sigma)(\lambda+1)(p+\lambda+2\sigma) h(\mu+1, \lambda+1, \sigma)
\end{aligned} \tag{93}
$$

Combining all these results, we obtain

$$
\begin{aligned}
&(\lambda+1)(p+\lambda+2\sigma)(\sigma+1)^2 h(\mu, \lambda, \sigma+1) \\
&\quad -(\lambda+1)(p+\lambda+2\sigma) i m_+ h(\mu, \lambda+1, \sigma) \\
&\quad = (\mu+1)(p+\mu+2\sigma)(\sigma+1)^2 h(\mu, \lambda, \sigma+1) \\
&\quad\quad +(\mu+1)(p+\mu+2\sigma) i m_- h(\mu+1, \lambda, \sigma)
\end{aligned} \tag{94}
$$

from which we get

$$
\begin{aligned}
&[(\lambda+1)(p+\lambda+2\sigma)-(\mu+1)(p+\mu+2\sigma)](\sigma+1)^2 h(\mu, \lambda, \sigma+1) \\
&\quad = (\lambda+1)(p+\lambda+2\sigma) i m_+ h(\mu, \lambda+1, \sigma) \\
&\quad\quad +(\mu+1)(p+\mu+2\sigma) i m_- h(\mu+1, \lambda, \sigma)
\end{aligned} \tag{95}
$$

By (85), (87) (under the conditions $\mu>0$ and $\lambda>0$) we have

$$
\begin{aligned}
&[(\lambda+1)(p+\lambda+2\sigma)-(\mu+1)(p+\mu+2\sigma)](\sigma+1)^2 h(\mu, \lambda, \sigma+1) \\
&\quad = (\lambda+1)(p+\lambda+2\sigma) i m_+ h(\mu, \lambda+1, \sigma) \\
&\quad\quad +(\mu+1)(p+\mu+2\sigma) i m_- h(\mu+1, \lambda, \sigma) \\
&\quad = h(\mu, \lambda, \sigma) \frac{(\lambda+1)(p+\lambda+2\sigma) i m_+(-i m_-)(\lambda+\mu+p+\sigma-1)}{(\lambda+1)[(\lambda+\mu+p+\sigma-1)(p+\lambda+2\sigma)-(\sigma+1)\mu]} \\
&\quad\quad + h(\mu, \lambda, \sigma) \frac{(\mu+1)(p+\mu+2\sigma) i m_-(+i m_+)(\lambda+\mu+p+\sigma-1)}{(\mu+1)[(\lambda+\mu+p+\sigma-1)(p+\mu+2\sigma)-(\sigma+1)\lambda]}
\end{aligned} \tag{96}
$$

The equation is nontrivial for $\lambda \neq \mu$; therefore,

$$(\lambda-\mu)(p+\mu+\lambda+2\sigma+1)(\sigma+1)^2 h(\mu, \lambda, \sigma+1)$$
$$= h(\mu, \lambda, \sigma) \frac{m_+ m_-(\lambda+\mu+p+\sigma-1)}{(\lambda+\mu+p+\sigma-1)(p+\lambda+2\sigma)-(\sigma+1)\mu}$$
$$\times \frac{-(p+\lambda+2\sigma)(\sigma+1)\lambda+(p+\mu+2\sigma)(\sigma+1)\mu}{(\lambda+\mu+p+\sigma-1)(p+\mu+2\sigma)-(\sigma+1)\lambda} \tag{97}$$

and so

$$h(\mu, \lambda, \sigma+1) = \frac{(-m_+ m_-)[(\lambda+\mu+p+\sigma-1)(p+\lambda+\mu+2\sigma)-(\sigma+1)\mu]}{(\lambda+\mu+p+\sigma-1)(p+\lambda+2\sigma)(\sigma+1)(p+\mu+\lambda+2\sigma+1)}$$
$$\times \frac{h(\mu, \lambda, \sigma)}{(\lambda+\mu+p+\sigma-1)(p+\mu+2\sigma)-(\sigma+1)\lambda} \tag{98}$$

under the conditions $\mu>0$, $\lambda>0$, $\mu \neq \lambda$.

Because of

$$[(\lambda+\mu+p+\sigma-1)(p+\lambda+2\sigma)-(\sigma+1)\mu]$$
$$\times[(\lambda+\mu+p+\sigma-1)(p+\mu+2\sigma)-(\sigma+1)\lambda]$$
$$=[(\lambda+p+\sigma-1)(p+\lambda+\mu+2\sigma)][(\mu+p+\sigma-1)(p+\lambda+\mu+2\sigma)] \tag{99}$$

we get

$$h(\mu, \lambda, \sigma+1) = \frac{(-m_+ m_-)(\lambda+\mu+p+\sigma-1)(p+\lambda+\mu+2\sigma)}{(p+\mu+\lambda+2\sigma+1)(\sigma+1)(\lambda+p+\sigma-1)(p+\lambda+\mu+2\sigma)}$$
$$\times \frac{h(\mu, \lambda, \sigma)}{(\mu+p+\sigma-1)(p+\lambda+\mu+2\sigma)} \tag{100}$$

A recursive expression for the coefficients in the representation series for the operator $\Phi^\dagger$ is therefore

$$h(\mu+1, \lambda, \sigma) = h(\mu, \lambda, \sigma) \frac{im_+(\lambda+\mu+p+\sigma-1)}{(\mu+1)(\mu+p+\sigma-1)(p+\mu+\lambda+2\sigma)} \quad \text{for} \quad \lambda>0 \tag{101}$$

$$h(\mu, \lambda+1, \sigma) = h(\mu, \lambda, \sigma) \frac{(-im_-)(\lambda+\mu+p+\sigma-1)}{(\lambda+1)(\lambda+p+\sigma-1)(p+\lambda+\mu+2\sigma)} \quad \text{for} \quad \mu>0 \tag{102}$$

$$h(\mu, \lambda, \sigma+1) = \frac{(-m_+m_-)(\lambda+\mu+p+\sigma-1)}{(\sigma+1)(\lambda+p+\sigma-1)(\mu+p+\sigma-1)} \times \frac{h(\mu, \lambda, \sigma)}{(p+\lambda+\mu+2\sigma)(p+\mu+\lambda+2\sigma+1)} \tag{103}$$

It remains the case $\lambda=\mu$. In (79) we set $\lambda \to \lambda+1$:

$$im_+h(\mu, \lambda+1, \sigma) - (\mu+1)(p+\mu+2\sigma)h(\mu+1, \lambda+1, \sigma) = (\sigma+1)^2 h(\mu, \lambda, \sigma+1) \tag{104}$$

and now let $\lambda=\mu$:

$$im_+h(\mu, \mu+1, \sigma) - (\mu+1)(p+\mu+2\sigma)h(\mu+1, \mu+1, \sigma) = (\sigma+1)^2 h(\mu, \mu, \sigma+1) \tag{105}$$

Analogously, we set in (80) $\mu \to \mu+1$ and get

$$-im_-h(\mu+1, \lambda, \sigma) - (\lambda+1)(p+\lambda+2\sigma)h(\mu+1, \lambda+1, \sigma) = (\sigma+1)^2 h(\mu, \lambda, \sigma+1) \tag{106}$$

and for $\mu=\lambda$

$$-im_-h(\mu+1, \mu, \sigma) - (\mu+1)(p+\mu+2\sigma)h(\mu+1, \mu+1, \sigma) = (\sigma+1)^2 h(\mu, \mu, \sigma+1) \tag{107}$$

It follows that

$$im_+h(\mu, \mu+1, \sigma) = -im_-h(\mu+1, \mu, \sigma) \tag{108}$$

and with $\lambda=\mu$ in (78) and with (87) we obtain

$$h(\mu+1, \mu+1, \sigma) = h(\mu+1, \mu, \sigma)\frac{(-im_-)(p+2\mu+\sigma)}{(\mu+1)(\mu+p+\sigma-1)(p+2\mu+2\sigma+1)} \tag{109}$$

for $\mu > 0$ and with (85) we arrive at

$$h(\mu+1, \mu, \sigma) = h(\mu, \mu, \sigma)\frac{im_+(2\mu+p+\sigma-1)}{(\mu+1)(\mu+p+\sigma-1)(p+2\mu+2\sigma)} \tag{110}$$

Therefore

$$h(\mu+1, \mu+1, \sigma) = \frac{im_+(2\mu+p+\sigma-1)(-im_-)(2\mu+p+\sigma)}{(\mu+1)(\mu+p+\sigma-1)(p+2\mu+2\sigma)(\mu+1)} \times \frac{h(\mu, \mu, \sigma)}{(\mu+p+\sigma-1)(p+2\mu+2\sigma+1)} \tag{111}$$

or

$$h(\mu, \mu, \sigma+1) = \frac{(-m_-m_+)(2\mu+p+\sigma-1)}{(\sigma+1)(\mu+p+\sigma-1)(\mu+p+\sigma-1)} \times \frac{h(\mu, \mu, \sigma)}{(p+2\mu+2\sigma)(p+2\mu+2\sigma+1)} \tag{112}$$

In the expansion of the series the first elements are given by direct computation from (68), (74), and (77) with the definition $h(0, 0, 0) = h$ and $h(\mu, \lambda, \sigma) = 0$ for $\mu < 0$ or $\lambda < 0$ or $\sigma < 0$. From (88) and (89) we get for $\mu = \sigma = \lambda = 0$

$$\begin{aligned}
h(1, 0, 0) &= h(0, 0, 0)\,\frac{im_+}{p} \\
h(0, 1, 0) &= h(0, 0, 0)\,\frac{-im_-}{p} \\
h(1, 1, 0) &= h(0, 1, 0)\,\frac{+im_+p}{(p-1)(p+1)} \\
h(1, 1, 0) &= h(1, 0, 0)\,\frac{-im_-p}{(p-1)(p+1)} \\
h(1, 1, 0) &= h(0, 0, 0)\,\frac{m_+m_-}{(p-1)(p+1)} \\
h(1, 1, 1) &= h(1, 1, 0)\,\frac{-m_+m_-(p+1)}{pp(p+2)(p+3)} \\
&= h(0, 0, 0)\,\frac{-m_+m_-m_+m_-}{(p-1)pp(p+2)(p+3)}
\end{aligned} \tag{113}$$

Herewith a recursive definition is given for the generator $\Phi^\dagger$ in (65) acting on the Lorentz vacuum for a momentum state of a massive scalar particle.

A state in rest is given by $m_+ = m_- = m$. From

$$h(\mu+1, \lambda, \sigma) = h(\mu, \lambda, \sigma)\,\frac{im_+(\lambda+\mu+\sigma+p-1)}{(\mu+1)(\mu+\sigma+p-1)(p+\mu+\lambda+2\sigma)} \tag{114}$$

we obtain

$$h(\mu+k, \lambda, \sigma) = h(\mu, \lambda, \sigma)\,\frac{(im_+)^k \mu!\,(\lambda+\mu+k-1+\sigma+p-1)!}{(\mu+k)!\,(\lambda+\mu-1+\sigma+p-1)!} \times \frac{(\mu-1+\sigma+p-1)!\,(p+\mu-1+\lambda+2\sigma)!}{(\mu+k-1+\sigma+p-1)!\,(\mu+k-1+\lambda+2\sigma+p)!} \tag{115}$$

or

$$h(k, \lambda, \sigma) = h(0, \lambda, \sigma)\,\frac{(im_+)^k(\lambda+k-1+\sigma+p-1)!}{k!\,(\lambda-1+\sigma+p-1)!} \times \frac{(-1+\sigma+p-1)!\,(p-1+\lambda+2\sigma)!}{(k-1+\sigma+p-1)!\,(k-1+\lambda+2\sigma+p)!} \tag{116}$$

We proceed in an analogous way for λ and σ and get as the final result

$$h(\mu, \lambda, \sigma) = h(0, 0, 0)\,\frac{(im_+)^\mu(-im_-)^\lambda\,(-m_+m_-)^\sigma(\lambda+\mu+\sigma+p-2)!}{\lambda!\,\mu!\,\sigma!\,(\lambda+\sigma+p-2)!\,(\mu+\sigma+p-2)!} \times \frac{1}{(\lambda+\mu+2\sigma+p-1)!} \tag{117}$$

for the coefficient $h(\mu, \lambda, \sigma)$ in the series (65) of a creation operator of a massive boson with

$$s = (n_1 + n_2 - n_3 - n_4) = 0 \tag{118}$$

4. SUMMARY AND CONCLUSIONS

We conclude that it is possible to describe quantum fields (possibly interacting) by ur theory. The key formulas in this description are explicit expressions for the dynamics (21) and (25). The ground state of the field theory can be calculated; it turns out to be unique, and it consists of infinitely many bits of information. We have constructed explicitly eigenstates of the energy and momentum operators for a given mass. It turns out that the spectrum of the energy and momentum operators is continuous, as should be expected, since arbitrary boosts of the center of mass of a given system allow any possible eigenvalue of these operators. Ur theory seems to be a promising concept for the description of quantum fields based on the multiple quantization of logic.

A. COMMUTATION RELATIONS

A.1. Application of Ur Generators on the Lorentz Vacuum

$$a_1|\omega\rangle = -ia_4^\dagger|\omega\rangle \qquad a_4|\omega\rangle = -ia_1^\dagger|\omega\rangle$$
$$a_2|\omega\rangle = ia_3^\dagger|\omega\rangle \qquad a_3|\omega\rangle = ia_2^\dagger|\omega\rangle \tag{A1}$$

$$\alpha_{14}|\omega\rangle = (-ip - \alpha_{14}^\dagger)|\omega\rangle \qquad \alpha_{23}|\omega\rangle = (ip - \alpha_{23}^\dagger|\omega\rangle$$
$$\alpha_{13}|\omega\rangle = \alpha_{24}^\dagger|\omega\rangle \qquad \alpha_{24}|\omega\rangle = \alpha_{13}^\dagger|\omega\rangle$$
$$(\alpha_{14} + \alpha_{14}^\dagger)|\omega\rangle = -ip|\omega\rangle \qquad (\alpha_{23} + \alpha_{23}^\dagger)|\omega\rangle = ip|\omega\rangle \tag{A2}$$

$$\tau_{14}|\omega\rangle = -i\alpha_{11}^\dagger|\omega\rangle \qquad \tau_{13}|\omega\rangle = +i\alpha_{12}^\dagger|\omega\rangle$$
$$\tau_{24}|\omega\rangle = -i\alpha_{12}^\dagger|\omega\rangle \qquad \tau_{23}|\omega\rangle = i\alpha_{22}^\dagger|\omega\rangle$$
$$\tau_{31}|\omega\rangle = -i\alpha_{34}^\dagger|\omega\rangle \qquad \tau_{41}|\omega\rangle = -i\alpha_{44}^\dagger|\omega\rangle$$
$$\tau_{32}|\omega\rangle = +i\alpha_{33}^\dagger|\omega\rangle \qquad \tau_{42}|\omega\rangle = +i\alpha_{34}^\dagger|\omega\rangle$$
$$\tau_{12}|\omega\rangle = +i\alpha_{13}^\dagger|\omega\rangle \qquad \tau_{21}|\omega\rangle = -i\alpha_{24}^\dagger|\omega\rangle$$
$$\tau_{43}|\omega\rangle = +i\alpha_{24}^\dagger|\omega\rangle \qquad \tau_{34}|\omega\rangle = -i\alpha_{13}^\dagger|\omega\rangle$$
$$v_1|\omega\rangle = v_4|\omega\rangle = -i\alpha_{14}^\dagger|\omega\rangle \qquad v_2|\omega\rangle = v_3|\omega\rangle = i\alpha_{23}^\dagger|\omega\rangle \tag{A3}$$

A.2. Commutation Relations between Momentum Operators and Ur Generators

$$[P_1 + iP_2, a_1^\dagger] = 0$$
$$[P_1 + iP_2, a_1] = -ia_2 - a_3^\dagger$$
$$[P_1 + iP_2, a_2^\dagger] = ia_1^\dagger + a_4$$
$$[P_1 + iP_2, a_2] = 0$$
$$[P_1 + iP_2, a_3^\dagger] = 0$$
$$[P_1 + iP_2, a_3] = ia_4 - a_1^\dagger$$
$$[P_1 + iP_2, a_4^\dagger] = -ia_3^\dagger + a_2$$
$$[P_1 + iP_2, a_4] = 0$$
$$[P_1 + iP_2, \alpha_{11}^\dagger] = 0$$
$$[P_1 + iP_2, \alpha_{11}] = 2(-i\alpha_{12} - \tau_{31})$$

$$
\begin{aligned}
&[P_1+iP_2,\alpha^\dagger_{12}]=i\alpha^\dagger_{11}+\tau_{14}\\
&[P_1+iP_2,\alpha_{12}]=-i\alpha_{22}-\tau_{32}\\
&[P_1+iP_2,\alpha^\dagger_{13}]=0\\
&[P_1+iP_2,\alpha_{13}]=-i\alpha_{23}+i\alpha_{14}-\nu_1-\nu_3-p\\
&[P_1+iP_2,\alpha^\dagger_{14}]=-i\alpha^\dagger_{13}+\tau_{12}\\
&[P_1+iP_2,\alpha_{14}]=-i\alpha_{24}-\tau_{34}\\
&[P_1+iP_2,\alpha^\dagger_{22}]=+2(i\alpha^\dagger_{12}+\tau_{24})\\
&[P_1+iP_2,\alpha_{22}]=0\\
&[P_1+iP_2,\alpha^\dagger_{23}]=+i\alpha^\dagger_{13}+\tau_{34}\\
&[P_1+iP_2,\alpha_{23}]=+i\alpha_{24}-\tau_{12}\\
&[P_1+iP_2,\alpha^\dagger_{24}]=+i\alpha^\dagger_{14}-i\alpha^\dagger_{23}+\nu_2+\nu_4+p\\
&[P_1+iP_2,\alpha_{24}]=0\\
&[P_1+iP_2,\nu_1]=-i\tau_{12}-\alpha^\dagger_{13}\\
&[P_1+iP_2,\nu_2]=+i\tau_{12}+\alpha_{24}\\
&[P_1+iP_2,\nu_3]=+i\tau_{34}-\alpha^\dagger_{13}\\
&[P_1+iP_2,\nu_4]=-i\tau_{34}+\alpha_{24}\\
&[P_1+iP_2,\tau_{12}]=0\\
&[P_1+iP_2,\tau_{21}]=+i(\nu_1-\nu_2)+\alpha_{14}-\alpha^\dagger_{23}\\
&[P_1+iP_2,\tau_{43}]=+i(\nu_4-\nu_3)+\alpha_{23}-\alpha^\dagger_{14}\\
&[P_1+iP_2,\tau_{34}]=0 \qquad\qquad \text{(A4)}\\
&[P_1-iP_2,a^\dagger_1]=+ia^\dagger_2-a_3\\
&[P_1-iP_2,a_1]=0\\
&[P_1-iP_2,a^\dagger_2]=0\\
&[P_1-iP_2,a_2]=-ia_1+a^\dagger_4\\
&[P_1-iP_2,a^\dagger_3]=-ia^\dagger_4-a_1\\
&[P_1-iP_2,a_3]=0\\
&[P_1-iP_2,a^\dagger_4]=0\\
&[P_1-iP_2,a_4]=+ia_3+a^\dagger_2
\end{aligned}
$$

$$
\begin{aligned}
&[P_1 - iP_2, \alpha_{11}^\dagger] = +2(+i\alpha_{12}^\dagger - \tau_{13})\\
&[P_1 - iP_2, \alpha_{11}] = 0\\
&[P_1 - iP_2, \alpha_{12}^\dagger] = +i\alpha_{22}^\dagger - \tau_{23}\\
&[P_1 - iP_2, \alpha_{12}] = -i\alpha_{11} + \tau_{41}\\
&[P_1 - iP_2, \alpha_{13}^\dagger] = +i\alpha_{23}^\dagger - i\alpha_{14}^\dagger - \nu_1 - \nu_3 - p\\
&[P_1 - iP_2, \alpha_{13}] = 0\\
&[P_1 - iP_2, \alpha_{14}^\dagger] = +i\alpha_{24}^\dagger - \tau_{43}\\
&[P_1 - iP_2, \alpha_{14}] = +i\alpha_{13} + \tau_{21}\\
&[P_1 - iP_2, \alpha_{22}^\dagger] = 0\\
&[P_1 - iP_2, \alpha_{22}] = +2(-i\alpha_{12} + \tau_{42})\\
&[P_1 - iP_2, \alpha_{23}^\dagger] = -i\alpha_{24}^\dagger - \tau_{21}\\
&[P_1 - iP_2, \alpha_{23}] = -i\alpha_{13} + \tau_{43}\\
&[P_1 - iP_2, \alpha_{24}^\dagger] = 0\\
&[P_1 - iP_2, \alpha_{24}] = -i\alpha_{14} + i\alpha_{23} + \nu_2 + \nu_4 + p\\
&[P_1 - iP_2, \nu_1] = +i\tau_{21} - \alpha_{13}\\
&[P_1 - iP_2, \nu_2] = -i\tau_{21} + \alpha_{24}^\dagger\\
&[P_1 - iP_2, \nu_3] = -i\tau_{43} - \alpha_{13}\\
&[P_1 - iP_2, \nu_4] = +i\tau_{43} + \alpha_{24}^\dagger\\
&[P_1 - iP_2, \tau_{12}] = +i(\nu_2 - \nu_1) - \alpha_{23} + \alpha_{14}^\dagger\\
&[P_1 - iP_2, \tau_{21}] = 0\\
&[P_1 - iP_2, \tau_{43}] = 0\\
&[P_1 - iP_2, \tau_{34}] = -i(\nu_4 - \nu_3) - \alpha_{14} + \alpha_{23}^\dagger
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
&[P_0 + P_3, a_1^\dagger] = +ia_1^\dagger + a_4\\
&[P_0 + P_3, a_1] = -ia_1 + a_4^\dagger\\
&[P_0 + P_3, a_2^\dagger] = 0\\
&[P_0 + P_3, a_2] = 0\\
&[P_0 + P_3, a_3^\dagger] = 0\\
&[P_0 + P_3, a_3] = 0
\end{aligned}
$$

$$
\begin{aligned}
&[P_0+P_3, a_4^\dagger]=+ia_4^\dagger+a_1\\
&[P_0+P_3, a_4]=-ia_4+a_1^\dagger\\
&[P_0+P_3, \alpha_{11}^\dagger]=+2(+i\alpha_{11}^\dagger+\tau_{14})\\
&[P_0+P_3, \alpha_{11}]=+2(-i\alpha_{11}+\tau_{41})\\
&[P_0+P_3, \alpha_{12}^\dagger]=+i\alpha_{12}^\dagger+\tau_{24}\\
&[P_0+P_3, \alpha_{12}]=-i\alpha_{12}+\tau_{42}\\
&[P_0+P_3, \alpha_{13}^\dagger]=+(i\alpha_{13}^\dagger+\tau_{34})\\
&[P_0+P_3, \alpha_{13}]=+(-i\alpha_{13}+\tau_{43})\\
&[P_0+P_3, \alpha_{14}^\dagger]=+2i\alpha_{14}^\dagger+v_1+v_4+p\\
&[P_0+P_3, \alpha_{14}]=-2i\alpha_{14}+v_1+v_4+p\\
&[P_0+P_3, \alpha_{22}^\dagger]=0\\
&[P_0+P_3, \alpha_{22}]=0\\
&[P_0+P_3, \alpha_{23}^\dagger]=0\\
&[P_0+P_3, \alpha_{23}]=0\\
&[P_0+P_3, \alpha_{24}^\dagger]=+i\alpha_{24}^\dagger+\tau_{21}\\
&[P_0+P_3, \alpha_{24}]=-i\alpha_{24}+\tau_{12}\\
&[P_0+P_3, v_1]=+\alpha_{14}+\alpha_{14}^\dagger\\
&[P_0+P_3, v_2]=0\\
&[P_0+P_3, v_3]=0\\
&[P_0+P_3, v_4]=+\alpha_{14}+\alpha_{14}^\dagger\\
&[P_0+P_3, \tau_{12}]=+i\tau_{12}+\alpha_{24}\\
&[P_0+P_3, \tau_{21}]=-i\tau_{21}+\alpha_{24}^\dagger\\
&[P_0+P_3, \tau_{43}]=+i\tau_{43}+\alpha_{13}\\
&[P_0+P_3, \tau_{34}]=-i\tau_{34}+\alpha_{13}^\dagger\\
&[P_0-P_3, a_1^\dagger]=0\\
&[P_0-P_3, a_1]=0\\
&[P_0-P_3, a_2^\dagger]=+ia_2^\dagger-a_3\\
&[P_0-P_3, a_2]=-ia_2-a_3^\dagger
\end{aligned}
\tag{A6}
$$

$$
\begin{aligned}
&[P_0 - P_3, a_3^\dagger] = +ia_3^\dagger - a_2 \\
&[P_0 - P_3, a_3] = -ia_3 - a_2^\dagger \\
&[P_0 - P_3, a_4^\dagger] = 0 \\
&[P_0 - P_3, a_4] = 0 \\
&[P_0 - P_3, \alpha_{11}^\dagger] = 0 \\
&[P_0 - P_3, \alpha_{11}] = 0 \\
&[P_0 - P_3, \alpha_{12}^\dagger] = +i\alpha_{12}^\dagger - \tau_{13} \\
&[P_0 - P_3, \alpha_{12}] = -i\alpha_{12} - \tau_{31} \\
&[P_0 - P_3, \alpha_{13}^\dagger] = +(i\alpha_{13}^\dagger - \tau_{12}) \\
&[P_0 - P_3, \alpha_{13}] = +(-i\alpha_{13} - \tau_{21}) \\
&[P_0 - P_3, \alpha_{14}^\dagger] = 0 \\
&[P_0 - P_3, \alpha_{14}] = 0 \\
&[P_0 - P_3, \alpha_{22}^\dagger] = +2(i\alpha_{22}^\dagger - \tau_{23}) \\
&[P_0 - P_3, \alpha_{22}] = +2(-i\alpha_{22} - \tau_{32}) \\
&[P_0 - P_3, \alpha_{23}^\dagger] = +(2i\alpha_{23}^\dagger - \nu_2 - \nu_3 - p) \\
&[P_0 - P_3, \alpha_{23}] = +(-2i\alpha_{23} - \nu_2 - \nu_3 - p) \\
&[P_0 - P_3, \alpha_{24}^\dagger] = +i\alpha_{24}^\dagger - \tau_{43} \\
&[P_0 - P_3, \alpha_{24}] = -i\alpha_{24} - \tau_{34} \\
&[P_0 - P_3, \nu_1] = 0 \\
&[P_0 - P_3, \nu_2] = -\alpha_{23} - \alpha_{23}^\dagger \\
&[P_0 - P_3, \nu_3] = -\alpha_{23} - \alpha_{23}^\dagger \\
&[P_0 - P_3, \nu_4] = 0 \\
&[P_0 - P_3, \tau_{12}] = -i\tau_{12} - \alpha_{13}^\dagger \\
&[P_0 - P_3, \tau_{21}] = +i\tau_{21} - \alpha_{13} \\
&[P_0 - P_3, \tau_{43}] = -i\tau_{43} - \alpha_{24}^\dagger \\
&[P_0 - P_3, \tau_{34}] = +i\tau_{34} - \alpha_{24} \qquad \text{(A7)} \\
&[P_1 + iP_2, \alpha_{11}^{\dagger\sigma}] = 0 \\
&[P_1 + iP_2, \alpha_{12}^{\dagger\rho}] = +\rho\alpha_{12}^{\dagger\rho-1}(i\alpha_{11}^\dagger + \tau_{14})
\end{aligned}
$$

$$[P_1+iP_2,\, \alpha_{13}^{\dagger\varphi}]=0$$
$$[P_1+iP_2,\, \alpha_{14}^{\dagger\mu}]=-\mu\alpha_{14}^{\dagger\mu-1}(i\alpha_{13}^{\dagger}-\tau_{12})$$
$$[P_1+iP_2,\, \alpha_{22}^{\dagger\zeta}]=+2\zeta\alpha_{22}^{\dagger\zeta-1}(i\alpha_{12}^{\dagger}+\tau_{24})$$
$$[P_1+iP_2,\, \alpha_{23}^{\dagger\lambda}]=+\lambda\alpha_{23}^{\dagger\lambda-1}(i\alpha_{13}^{\dagger}+\tau_{34})$$
$$[P_1+iP_2,\, \alpha_{24}^{\dagger\beta}]=+\beta\alpha_{24}^{\dagger\beta-1}[i\alpha_{14}^{\dagger}-i\alpha_{23}^{\dagger}+(\nu_2+\nu_4+p+\beta-1)]$$
$$[P_1+iP_2,\, \alpha_{14}^{\mu}]=+\mu\alpha_{14}^{\mu-1}(+i\alpha_{13}+\tau_{21})$$
$$[P_1+iP_2,\, \alpha_{13}^{\varphi}]=\varphi\alpha_{13}^{\varphi-1}(-i\alpha_{23}+i\alpha_{14}-\nu_1-\nu_3+\varphi-1-p)$$
$$[P_1+iP_2,\, \alpha_{24}^{\beta}]=0$$
$$[P_1+iP_2,\, \alpha_{23}^{\lambda}]=+\lambda\alpha_{23}^{\lambda-1}(-i\alpha_{13}+\tau_{43})$$
$$[P_1+iP_2,\, \nu_1^k]=[\nu_1^k-(\nu_1-1)^k](-i\tau_{12}-\alpha_{13}^{\dagger})$$
$$[P_1+iP_2,\, \nu_2^k]=[(\nu_2+1)^k-\nu_2^k](+i\tau_{12}+\alpha_{24})$$
$$[P_1+iP_2,\, \nu_3^k]=[\nu_3^k-(\nu_3-1)^k](+i\tau_{34}-\alpha_{13}^{\dagger})$$
$$[P_1+iP_2,\, \nu_4^k]=[(\nu_4+1)^k-\nu_4^k](-i\tau_{34}+\alpha_{24})$$
$$[P_1+iP_2,\, \tau_{12}^k]=0$$
$$[P_1+iP_2,\, \tau_{21}^k]=ik\tau_{21}^{k-1}(\nu_1-\nu_2-k+1)+[(\tau_{21}+1)^k-\tau_{21}^k]\alpha_{14} -[\tau_{21}^k-(\tau_{21}-1)^k]\alpha_{23}^{\dagger}$$
$$[P_1+iP_2,\, \tau_{34}^k]=0$$
$$[P_1+iP_2,\, \tau_{43}^k]=ik\tau_{43}^{k-1}(\nu_4-\nu_3+k-1)+[(\tau_{43}+1)^k-\tau_{43}^k]\alpha_{23} -[\tau_{43}^k-(\tau_{43}-1)^k]\alpha_{14}^{\dagger}$$
$$[P_1+iP_2,\, \alpha_{14}^{\dagger\mu}\alpha_{23}^{\dagger\lambda}]=+\alpha_{14}^{\dagger\mu-1}\alpha_{23}^{\dagger\lambda-1} \times[\mu\lambda\alpha_{13}^{\dagger}+\lambda\alpha_{14}^{\dagger}(i\alpha_{13}^{\dagger}+\tau_{34})-\mu\alpha_{23}^{\dagger}(i\alpha_{13}^{\dagger}-\tau_{12})]$$
$$[P_1+iP_2,\, \alpha_{13}^{\dagger\varphi}\alpha_{24}^{\dagger\beta}]=+\beta\alpha_{13}^{\dagger\varphi}\alpha_{24}^{\dagger\beta-1} \times[i\alpha_{14}^{\dagger}-i\alpha_{23}^{\dagger}+(\nu_2+\nu_4+p+\beta-1)] \qquad \text{(A8)}$$
$$[P_1-iP_2,\, \alpha_{11}^{\dagger\sigma}]=+2\sigma\alpha_{11}^{\dagger\sigma-1}(i\alpha_{12}^{\dagger}-\tau_{13})$$
$$[P_1-iP_2,\, \alpha_{12}^{\dagger\rho}]=+\rho\alpha_{12}^{\dagger\rho-1}(i\alpha_{22}^{\dagger}-\tau_{23})$$
$$[P_1-iP_2,\, \alpha_{13}^{\dagger\varphi}]=+\varphi\alpha_{13}^{\dagger\varphi-1}[i\alpha_{23}^{\dagger}-i\alpha_{14}^{\dagger}-(\nu_1+\nu_3+p+\varphi-1)]$$
$$[P_1-iP_2,\, \alpha_{14}^{\dagger\mu}]=+\mu\alpha_{14}^{\dagger\mu-1}(i\alpha_{24}^{\dagger}-\tau_{43})$$
$$[P_1-iP_2,\, \alpha_{22}^{\dagger\zeta}]=0$$

$$[P_1 - iP_2, \alpha_{23}^{\dagger\lambda}] = -\lambda\alpha_{23}^{\dagger\lambda-1}(i\alpha_{24}^\dagger + \tau_{21})$$

$$[P_1 - iP_2, \alpha_{24}^{\dagger\beta}] = 0$$

$$[P_1 - iP_2, \alpha_{14}^{\mu}] = +\mu(-i\alpha_{24} - \tau_{34})\alpha_{14}^{\mu-1}$$

$$[P_1 - iP_2, \alpha_{13}^{\varphi}] = 0$$

$$[P_1 - iP_2, \alpha_{24}^{\beta}] = \beta\alpha_{24}^{\beta-1}(-i\alpha_{14} + i\alpha_{23} + \nu_2 + \nu_4 - \beta + 1 + p)$$

$$[P_1 - iP_2, \alpha_{23}^{\lambda}] = -\lambda(-i\alpha_{24} + \tau_{12})\alpha_{23}^{\lambda-1}$$

$$[P_1 - iP_2, \nu_1^k] = [(\nu_1 + 1)^k - \nu_1^k](+i\tau_{21} - \alpha_{13})$$

$$[P_1 - iP_2, \nu_2^k] = [\nu_2^k - (\nu_2 - 1)^k](-i\tau_{21} + \alpha_{24}^\dagger)$$

$$[P_1 - iP_2, \nu_3^k] = [(\nu_3 + 1)^k - \nu_3^k](-i\tau_{43} - \alpha_{13})$$

$$[P_1 - iP_2, \nu_4^k] = [\nu_4^k - (\nu_4 - 1)^k](+i\tau_{43} + \alpha_{24}^\dagger)$$

$$[P_1 - iP_2, \tau_{12}^k] = ik\tau_{12}^{k-1}(\nu_2 - \nu_1 - k + 1)$$
$$+ [\tau_{12}^k - (\tau_{12} - 1)^k]\alpha_{14}^\dagger - [(\tau_{12} + 1)^k - \tau_{12}^k]\alpha_{23}$$

$$[P_1 - iP_2, \tau_{21}^k] = 0$$

$$[P_1 - iP_2, \tau_{34}^k] = ik\tau_{34}^{k-1}(\nu_3 - \nu_4 + k - 1)$$
$$-[(\tau_{34} + 1)^k - \tau_{34}^k]\alpha_{14} + [\tau_{34}^k - (\tau_{34} - 1)^k]\alpha_{23}^\dagger$$

$$[P_1 - iP_2, \tau_{43}^k] = 0$$

$$[P_1 - iP_2, \alpha_{14}^{\dagger\mu}\alpha_{23}^{\dagger\lambda}] = -\alpha_{14}^{\dagger\mu-1}\alpha_{23}^{\dagger\lambda-1}$$
$$\times[\mu\lambda\alpha_{24}^\dagger - \lambda\alpha_{14}^\dagger(i\alpha_{24}^\dagger + \tau_{21}) + \mu\alpha_{23}^\dagger(i\alpha_{24}^\dagger - \tau_{43})]$$

$$[P_1 - iP_2, \alpha_{13}^{\dagger\varphi}\alpha_{24}^{\dagger\beta}] = +\varphi\alpha_{13}^{\dagger\varphi-1}\alpha_{24}^{\dagger\beta}$$
$$\times[i\alpha_{23}^\dagger - i\alpha_{14}^\dagger - (\nu_1 + \nu_3 + p + \varphi - 1)] \qquad \text{(A9)}$$

$$[P_0 + P_3, \alpha_{11}^{\dagger\sigma}] = +2\sigma\alpha_{11}^{\dagger\sigma-1}(i\alpha_{11}^\dagger + \tau_{14})$$

$$[P_0 + P_3, \alpha_{12}^{\dagger\rho}] = +\rho\alpha_{12}^{\dagger\rho-1}(i\alpha_{12}^\dagger + \tau_{24})$$

$$[P_0 + P_3, \alpha_{13}^{\dagger\varphi}] = +\varphi\alpha_{13}^{\dagger\varphi-1}(i\alpha_{13}^\dagger + \tau_{34})$$

$$[P_0 + P_3, \alpha_{14}^{\dagger\mu}] = +\mu\alpha_{14}^{\dagger\mu-1}(2i\alpha_{14}^\dagger + \nu_1 + \nu_4 + p + \mu - 1)$$

$$[P_0 + P_3, \alpha_{22}^{\dagger\zeta}] = 0$$

$$[P_0 + P_3, \alpha_{23}^{\dagger\lambda}] = 0$$

$$[P_0 + P_3, \alpha_{24}^{\dagger\beta}] = +\beta\alpha_{24}^{\dagger\beta-1}(i\alpha_{24}^\dagger + \tau_{21})$$

$$
\begin{aligned}
[P_0+P_3,\ \alpha_{14}^{\mu}] &= \mu(-2i\alpha_{14}+\nu_1+\nu_4+p+\mu-1)\alpha_{14}^{\mu-1}\\
&= \mu\alpha_{14}^{\mu-1}(-2i\alpha_{14}+\nu_1+\nu_4+p-\mu+1)\\
[P_0+P_3,\ \alpha_{13}^{\varphi}] &= +\varphi(-i\alpha_{13}+\tau_{43})\alpha_{13}^{\varphi-1}\\
[P_0+P_3,\ \alpha_{24}^{\beta}] &= \beta\alpha_{24}^{\beta-1}(-i\alpha_{24}+\tau_{12})\\
[P_0+P_3,\ \alpha_{23}^{\lambda}] &= 0\\
[P_0+P_3,\ \nu_1^{k}] &= [(\nu_1+1)^k-\nu_1^k]\alpha_{14}+[\nu_1^k-(\nu_1-1)^k]\alpha_{14}^{\dagger}\\
[P_0+P_3,\ \nu_2^{k}] &= 0\\
[P_0+P_3,\ \nu_3^{k}] &= 0\\
[P_0+P_3,\ \nu_4^{k}] &= [(\nu_4+1)^k-\nu_4^k]\alpha_{14}+[\nu_4^k-(\nu_4-1)^k]\alpha_{14}^{\dagger}\\
[P_0+P_3,\ \tau_{12}^{k}] &= +k\tau_{12}^{k-1}i\tau_{12}+[(\tau_{12}+1)^k-\tau_{12}^k]\alpha_{24}\\
[P_0+P_3,\ \tau_{21}^{k}] &= -k\tau_{21}^{k-1}i\tau_{21}+[\tau_{21}^k-(\tau_{21}-1)^k]\alpha_{24}^{\dagger}\\
[P_0+P_3,\ \tau_{43}^{k}] &= +k\tau_{43}^{k-1}i\tau_{43}+[(\tau_{43}+1)^k-\tau_{43}^k]\alpha_{13}\\
[P_0+P_3,\ \tau_{34}^{k}] &= -k\tau_{34}^{k-1}i\tau_{34}+[\tau_{34}^k-(\tau_{34}-1)^k]\alpha_{13}^{\dagger}\\
[P_0+P_3,\ \alpha_{14}^{\dagger\mu}\alpha_{23}^{\dagger\lambda}] &= +\mu\alpha_{14}^{\dagger\mu-1}\alpha_{23}^{\dagger\lambda}(2i\alpha_{14}^{\dagger}+\nu_1+\nu_4+p+\mu-1)\\
[P_0+P_3,\ \alpha_{13}^{\dagger\varphi}\alpha_{24}^{\dagger\beta}] &= +\alpha_{13}^{\dagger\varphi-1}\alpha_{24}^{\dagger\beta-1}[\varphi\beta\alpha_{23}^{\dagger}+\varphi\alpha_{24}^{\dagger}(i\alpha_{13}^{\dagger}+\tau_{34})\\
&\quad+\beta\alpha_{13}^{\dagger}(i\alpha_{24}^{\dagger}+\tau_{21})]
\end{aligned}
\tag{A10}
$$

$$
\begin{aligned}
[P_0-P_3,\ \alpha_{11}^{\dagger\sigma}] &= 0\\
[P_0-P_3,\ \alpha_{12}^{\dagger\rho}] &= +\rho\alpha_{12}^{\dagger\rho-1}(i\alpha_{12}^{\dagger}-\tau_{13})\\
[P_0-P_3,\ \alpha_{13}^{\dagger\varphi}] &= +\varphi\alpha_{13}^{\dagger\varphi-1}(i\alpha_{13}^{\dagger}-\tau_{12})\\
[P_0-P_3,\ \alpha_{14}^{\dagger\mu}] &= 0\\
[P_0-P_3,\ \alpha_{22}^{\dagger\zeta}] &= +2\zeta\alpha_{22}^{\dagger\zeta-1}(i\alpha_{22}^{\dagger}-\tau_{23})\\
[P_0-P_3,\ \alpha_{23}^{\dagger\lambda}] &= +\lambda\alpha_{23}^{\dagger\lambda-1}[2i\alpha_{23}^{\dagger}-(\nu_2+\nu_3+p+\lambda-1)]\\
[P_0-P_3,\ \alpha_{24}^{\dagger\beta}] &= +\beta\alpha_{24}^{\dagger\beta-1}(+i\alpha_{24}^{\dagger}-\tau_{43})\\
[P_0-P_3,\ \alpha_{14}^{\mu}] &= 0\\
[P_0-P_3,\ \alpha_{13}^{\varphi}] &= +\varphi(-i\alpha_{13}-\tau_{21})\alpha_{13}^{\varphi-1}\\
[P_0-P_3,\ \alpha_{24}^{\beta}] &= +\beta(-i\alpha_{24}-\tau_{34})\alpha_{24}^{\beta-1}\\
[P_0-P_3,\ \alpha_{23}^{\lambda}] &= +\lambda(-2i\alpha_{23}-\nu_2-\nu_3-p)\alpha_{23}^{\lambda-1}\\
[P_0-P_3,\ \nu_1^{k}] &= 0
\end{aligned}
$$

$$[P_0-P_3,\, \nu_2^k]=+[(\nu_2+1)^k-\nu_2^k](-\alpha_{23})+[\nu_2^k-(\nu_2-1)^k](-\alpha_{23}^\dagger)$$
$$[P_0-P_3,\, \nu_3^k]=+[(\nu_3+1)^k-\nu_3^k](-\alpha_{23})+[\nu_3^k-(\nu_3-1)^k](-\alpha_{23}^\dagger)$$
$$[P_0-P_3,\, \nu_4^k]=0$$
$$[P_0-P_3,\, \tau_{12}^k]=-k\tau_{12}^{k-1}i\tau_{12}-[\tau_{12}^k-(\tau_{12}-1)^k]\alpha_{13}^\dagger$$
$$[P_0-P_3,\, \tau_{21}^k]=+k\tau_{21}^{k-1}i\tau_{21}-[(\tau_{21}+1)^k-\tau_{21}^k]\alpha_{13}$$
$$[P_0-P_3,\, \tau_{34}^k]=+k\tau_{34}^{k-1}i\tau_{34}-[(\tau_{34}+1)^k-\tau_{34}^k]\alpha_{24}$$
$$[P_0-P_3,\, \tau_{43}^k]=-k\tau_{43}^{k-1}i\tau_{43}-[\tau_{43}^k-(\tau_{43}-1)^k]\alpha_{24}^\dagger$$
$$[P_0-P_3,\, \alpha_{14}^{\dagger\mu}\alpha_{23}^{\dagger\lambda}]=\lambda\alpha_{14}^{\dagger\mu}\alpha_{23}^{\dagger\lambda-1}[+2i\alpha_{23}^\dagger-(\nu_2+\nu_3+p+\lambda-1)]$$
$$[P_0-P_3,\, \alpha_{13}^{\dagger\varphi}\alpha_{24}^{\dagger\beta}]=\alpha_{13}^{\dagger\varphi-1}\alpha_{24}^{\dagger\beta-1}[-\varphi\beta\alpha_{14}^\dagger+\varphi\alpha_{24}^\dagger(+i\alpha_{13}^\dagger-\tau_{12})$$
$$+\beta\alpha_{13}^\dagger(+i\alpha_{24}^\dagger+\tau_{43})] \qquad \text{(A11)}$$

REFERENCES

Bekenstein, J. D. (1973). *Physical Review D*, **7**, 2333–2346.

Castell, L. (1975). Quantum theory of simple alternatives, in *Quantum Theory of the Structures of Time and Space*, L. Castell, M. Drieschner, and C. F. v. Weizsäcker, eds.

Drieschner, M., Görnitz, Th., and v. Weizsäcker, C. F. (1988). *International Journal of Theoretical Physics*, **27**, 289–306.

Finkelstein, D. (1969). *Physical Review*, **184**, 1261.

Görnitz, Th. (1986). *International Journal of Theoretical Physics*, **25**, 897.

Görnitz, Th. (1988). *International Journal of Theoretical Physics*, **27**, 527.

Görnitz, Th. (1991). The role of para-Bose statistics in making abstract quantum theory concrete, in *Symmetries in Science*, *V*, B. Gruber, ed., Plenum Press, New York.

Graudenz, D. (1990). *Journal of Mathematical Physics*, **31**, 1652.

Green, H. S. (1953). *Physical Review*, **90**, 270.

Hawking, S. W. (1975). *Communications in Mathematical Physics*, **43**, 199–220.

Heidenreich, W. (1981). Die dynamischen Gruppen $SO(3,2)$ und $SO(4,2)$ als Raum-Zeit-Gruppen von Elementarteilchen, Thesis, TU München.

Jacob, P. (1977). Konform invariante Theorie exklusiver Elementarteilchen-Streuungen bei grossen Winkeln, Thesis, MPI Starnberg.

Künemund, Th. (1982). Dynamische Symmetrien in der Elementarteilchenphysik, Diploma Thesis, TU München.

Künemund, Th. (1985). Die Darstellungen der symplektischen und konformen Superalgebren, Thesis, TU München.

Scheibe, E., Süssmann, G., and v. Weizsäcker, C. F. (1958). *Zeitschrift für Naturforschung*, **13a**, 705.

v. Weizsäcker, C. F. (1955). *Naturwissenschaften*, **42**, 521–529, 545–555.

v. Weizsäcker, C. F. (1985). *Aufbau der Physik*, Carl Hanser Verlag, Munich.